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VARIATIONAL PRINCIPLES OF THE THEORY OF THE LIMITING EQUILIBRIUM OF MEDIA WITH DIFFERENT STRENGTHS†

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The simplest phenomenological models of materials having different resistances to tension and compression are constructed using a rheological method supplemented by a new element – a rigid contact. The problems of the solvability of static boundary-value problems for small deformations of the medium are considered within the framework of the regularized model. A generalization of the static and kinematic theorems of the theory of limiting equilibrium is given. The upper bound of the limiting load and the angle of emergence of the linear zone of deformation localization in the problem of the rupture of a cylindrical specimen with a radial notch when there is pressure on the sides of the notch is obtained as an example of the application of these theorems. © 2004 Elsevier Ltd. All rights reserved.

1. THE MATHEMATICAL MODEL

Rheological models of materials, having different resistance to tension and compression (granular and porous media: soils, rocks, concretes, graphites, etc.), are constructed using an auxiliary element – a rigid contact [1]. Such a system is illustrated in Fig. 1, for the simplest model, taking into account the connectivity of the medium. Such a medium does not deform under compressive stresses or tensile stresses less than the cohesion coefficient σ_0 (the yield point of the plastic element). The attainment of the value of σ_0 corresponds to limiting equilibrium, in which the deformation can be an arbitrary positive quantity. Stresses above this limit are impossible. The constitutive equations of the uniaxial deformation for monotonic loading without unloading result in the following system

$$\sigma \le \sigma_0, \quad \varepsilon \ge 0 \quad (\sigma - \sigma_0)\varepsilon = 0$$

This system is equivalent to the variational inequalities

$$(\sigma - \sigma_0)(\tilde{\varepsilon} - \varepsilon) \le 0, \quad \varepsilon, \tilde{\varepsilon} \ge 0; \quad (\tilde{\sigma} - \sigma)\varepsilon \le 0, \quad \sigma, \tilde{\sigma} \le \sigma_0$$

(ε and σ are variables), each of which allows of the potential representation

$$\sigma \in \partial\varphi(\varepsilon), \quad \varepsilon \in \partial\psi(\sigma) \tag{1.1}$$

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‡Veniamin Petrovich Myasnikov (1936–2004), Academician, a prominent scientist in the field of mechanics. He was Director of the Institute for Automation and Control of the Far-East Branch of Russian Academy of Sciences, founder and first Head of Department of Computational Mechanics in the Mechanics and Mathematics faculty of Moscow State University, and Head of Department of Computational Geophysics at the M. V. Keldysh Institute of Applied Mathematics. He developed the theory of the motion of a gas when it seeps through a layer of granular material in a chemical reactor. He also made important contributions to the development of direct variational methods in the theory of rigid plastic media. He developed rigorous mathematical methods in the theory of convective flows, a model of convection inside the Earth, and also a model of the transition layers which arise in the evolution of the Earth. He suggested a new quasi-linear modification of Hooke's law, in which the moduli of elasticity depend on the invariants of the strain tensor. He developed a fundamentally new approach to the theory of plasticity based on methods of the theory of gauge fields. He trained numerous pupils and followers. More than 30 Candidate dissertations were prepared by students under his supervision, five of whom became Doctors of Science.

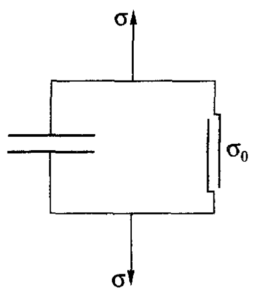

Fig. 1

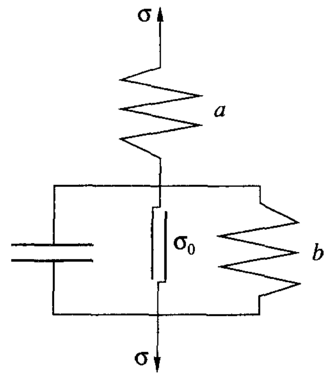

Fig. 2

Here $\varphi = \sigma_0\varepsilon + \delta_G(\varepsilon)$ and $\psi = \delta_K(\sigma - \sigma_0)$ are the potentials of the stresses and strains. Functions vanishing on the cones $C = \{\varepsilon \geqslant 0\}$ and $K = \{\sigma \leqslant 0\}$ and infinite outside these cones are denoted by δ; δ serves to notate the subdifferential

$$\partial\varphi(\varepsilon) = \{\sigma | \varphi(\tilde{\varepsilon}) - \varphi(\varepsilon) \geq \sigma(\tilde{\varepsilon} - \varepsilon) \ \ \forall\tilde{\varepsilon}\}$$

which is the set of slopes of linear functions, the graphs of which pass through the point $(\varepsilon, \varphi(\varepsilon))$ and lie below the graph of the function φ.

The extension to the case of the three-dimensional stress–strain state is constructed on the basis of Eq. (1.1). For this a symmetrical cohesion tensor σ_0, a convex and closed cone C with vertex at the origin of the six-dimensional space of the strain tensors, or an analogous cone K in the space of the stress tensors are given. If one of the cones is known, then the second one can be obtained as the conjugate

$$K = \{\sigma | \sigma : \varepsilon \leq 0 \ \ \forall\varepsilon \in C\}, \quad C = \{\varepsilon | \sigma : \varepsilon \leq 0 \ \ \forall\sigma \in K\}$$

(the colon denotes the convolution operation). The cone K and the tensor σ_0 must satisfy condition $-\sigma_0 \in K$, which indicates the admissibility of the natural stress-free state of the medium. The potentials $\varphi(\varepsilon)$ and $\psi(\sigma)$ are obtained automatically by replacing the scalar quantities by tensor ones and the product by convolution. These potentials are double convex functions, i.e. they are expressed in terms of one another using Young's transformation

$$\varphi(\varepsilon) = \sup_{\sigma}\{\sigma : \varepsilon - \psi(\sigma)\}, \quad \psi(\sigma) = \sup_{\varepsilon}\{\sigma : \varepsilon - \varphi(\varepsilon)\}$$

The constitutive equations (1.1), which describe the behaviour of the connected granular medium with rigid grains, are to a certain extent incorrect: they do not enable one to find the deformed state for specified stresses uniquely or to find the stress for specified deformations. A model of a granular medium with elastic grains serves as the regularization. The rheological diagram of this model is presented in Fig. 2.

Suppose a and b are symmetrical positive definite fourth-rank tensors, comprised of the moduli of elasticity of the regularizing elements and let d for now be an arbitrary tensor with similar properties. If the loading is monotonic, then the constitutive equations can be represented in the form (1.1) using convex and differentiating potentials, which are formulated in terms of the projections of the strain and stress tensors onto the respective cones. It is well known [2] that the projection $\pi_d(\varepsilon)$ of the tensor ε onto the cone C with norm $|\varepsilon|_d = \sqrt{\varepsilon : d : \varepsilon}$ is the unique solution of the variational inequality

$$(\varepsilon - \pi_d(\varepsilon)) : d : (\tilde{\varepsilon} - \pi_d(\varepsilon)) \leq 0, \quad \pi_d(\varepsilon), \tilde{\varepsilon} \in C$$

or the equivalent system

$$(\varepsilon - \pi_d(\varepsilon)) : d : \pi_d(\varepsilon) = 0 \quad (\varepsilon - \pi_d(\varepsilon)) : d : \tilde{\varepsilon} \leq 0 \tag{1.2}$$

In turns out that, like the case of orthogonal subspaces, each tensor can be represented in the form of the sum of its projections onto the conjugate cones

$$\varepsilon = \pi_d(\varepsilon) + d^{-1} : \Pi_{d^{-1}}(d : \varepsilon) \tag{1.3}$$

where $\Pi_{d^{-1}}(\sigma)$ is the projection σ onto K with norm $|\sigma|_{d^{-1}}$, associated with the inverse tensor d^{-1}.

Actually, the inequality in system (1.2) exactly means that the tensor $\sigma = d : (\varepsilon - \pi_d(\varepsilon)$ belongs to the cone K, conjugate to C. Furthermore, the following equation holds

$$\sigma : d^{-1} : (d : \varepsilon - \sigma) = 0$$

by virtue of Eq. (1.2), and since $\pi_d(\varepsilon) \in$ C, then by definition of the conjugate cone the following inequality is satisfied

$$\tilde{\sigma} : d^{-1} : (d : \varepsilon - \sigma) \leq 0, \quad \tilde{\sigma} \in K$$

Hence $\sigma = \Pi_{d^{-1}}(d : \varepsilon)$, which corresponds to relation (1.3).

The identity

$$|\varepsilon|_d^2 = |\pi_d(\varepsilon)|_d^2 + |\varepsilon - \pi_d(\varepsilon)|_d^2 = |\pi_d(\varepsilon)|_d^2 + |\Pi_{d^{-1}}(d : \varepsilon)|_{d^{-1}}^2 \tag{1.4}$$

is proved, taking into account Eqs (1.2) and (1.3).

Note that in the more general case when C is an arbitrary convex set, the term

$$|\varepsilon - \pi_d(\varepsilon)|_d^2 = \inf_{\xi \in C} |\varepsilon - \xi|_d^2$$

is a differential convex function [3], the derivative of which is equal to $2d : (\varepsilon - \pi_d(\varepsilon))$. Hence the function $|\pi_d(\varepsilon)|_d^2$ is also differentiable, where

$$\frac{\partial}{\partial \varepsilon}|\pi_d(\varepsilon)|_d^2 = 2d : \pi_d(\varepsilon) \tag{1.5}$$

Note also that the projector onto the cone is positive homogeneous, i.e.

$$\pi_d(\lambda\varepsilon) = \lambda\pi_d(\varepsilon), \quad \lambda \geq 0$$

This follows from Eq. (1.2).

Formulae (1.3) and (1.4) enables us to consider only one of the cones without calculating its conjugate in explicit form. A medium possessing elastic properties, according to the rheological model (Fig. 2), is characterized by the system

$$\varepsilon = \varepsilon_a + \varepsilon_b, \quad \sigma = a : \varepsilon_a \quad (\sigma - \sigma_0 - b : \varepsilon_b) : (\tilde{\varepsilon} - \varepsilon_b) \leq 0, \quad \varepsilon_b, \tilde{\varepsilon} \in C$$

From this system it follows that

$$\varepsilon = a^{-1} : \sigma + \pi_b(b^{-1} : (\sigma - \sigma_0)) \tag{1.6}$$

Using (1.5) we can obtain the potential of the deformations

$$\psi = \frac{1}{2}(|\sigma|_{a^{-1}}^2 + |\pi_b(b^{-1} : (\sigma - \sigma_0))|_b^2) = \frac{1}{2}\left(|\sigma|_{a^{-1}}^2 + |\sigma - \sigma_0|_{b^{-1}}^2 - |\Pi_{b^{-1}}(\sigma - \sigma_0)|_{b^{-1}}^2\right) \tag{1.7}$$

The coincidence of the norm of the tensor and the norm of its projection onto the cone occurs only for the elements of the cone by virtue of identity (1.4). Consequently, only in the case when $\sigma - \sigma_0 \in K$ is the potential (1.7) equal to the deformation energy of the elastic element with the tensor of the coefficients a. In the limit as $b \to 0$, this potential tends to

$$\psi_a = |\sigma|_{a^{-1}}^2/2 + \delta_K(\sigma - \sigma_0)$$

and as $a \to \infty$ it tends to

$$\psi_b = |\pi_b(b^{-1} : (\sigma - \sigma_0))|_b^2/2$$

The dual potential of the stresses, that is Young's transformation of the function $\psi(\sigma)$, equals

$$\varphi = \frac{1}{2}\sup_{\sigma}\inf_{\xi\in C}\left\{2\sigma:\varepsilon - |\sigma|^2_{a^{-1}} - |\sigma-\sigma_0|^2_{b^{-1}} + |\sigma-\sigma_0-b:\xi|^2_{b^{-1}}\right\}$$

An exact upper bound can be calculated after interchanging sup and inf. It is reached for the tensor ξ for fixed $\sigma = a:(\varepsilon-\xi)$, and hence

$$\varphi = \frac{1}{2}\inf_{\xi\in C}\{|\varepsilon-\xi|^2_a + |\xi|^2_b + 2\sigma_0:\xi\}$$

The separation of the total square relative to the tensor ξ results in the equation

$$\varphi = \frac{1}{2}(|\varepsilon|^2_a - |a:\varepsilon-\sigma_0|^2_{d^{-1}}) + \frac{1}{2}\inf_{\xi\in C}|d^{-1}:(a:\varepsilon-\sigma_0)-\xi|^2_d$$

where $d = a + b$. Hence

$$\varphi = \frac{1}{2}(|\varepsilon|^2_a - |\pi_d(d^{-1}:(a:\varepsilon-\sigma_0))|^2_d) = \frac{1}{2}(|\varepsilon|^2_a - |a:\varepsilon-\sigma_0|^2_{d^{-1}} + |\Pi_{d^{-1}}(a:\varepsilon-\sigma_0)|^2_{d^{-1}}) \tag{1.8}$$

According to Eq. (1.8) the potential of the stresses is a differentiable function. By virtue of relation (1.5)

$$\sigma = a:\{\varepsilon - \pi_d(d^{-1}:(a:\varepsilon-\sigma_0))\} \tag{1.9}$$

Taking the limit with respect to b as $b\to 0$ the potential of the stresses for the medium, whose rheological model contains one elastic element

$$\varphi_a = \frac{1}{2}(|\varepsilon|^2_a - |\pi_a(\varepsilon - a^{-1}:\sigma_0)|^2_a)$$

can be obtained. This potential is dual to $\psi_a(\sigma)$. Similarly, as $a\to\infty$, we obtain the potential

$$\varphi_b = |\varepsilon|^2_b/2 + \sigma_0:\varepsilon + \delta_C(\varepsilon)$$

dual to $\psi_b(\sigma)$.

A closed mathematical model to describe the equilibrium of the medium with potentials $\varphi(\varepsilon)$ and $\psi(\sigma)$ for small deformations is formed by the constitutive equations in the form (1.6) or (1.9), supplemented by the conditions of equilibrium and geometric constraints

$$\nabla\cdot\sigma + f = 0,\quad 2\varepsilon(u) = \nabla u + (\nabla u)^* \tag{1.10}$$

Hence u is the vector of the displacements and f is the vector of the body forces. The asterisk denotes the transposition of the tensor, and the generally accepted notation of tensor analysis is used.

2. THE EXISTENCE OF SOLUTIONS

Let Ω be the space occupied by the medium or a plane domain with a boundary Γ consisting of two disjoint parts Γ_u and Γ_σ with no displacements on the first and a specified distributed load on the second (ν is the vector of the normal):

$$u = 0 \text{ on } \Gamma_u,\quad \sigma\cdot\nu = q \text{ on } \Gamma_\sigma \tag{2.1}$$

The problem is to determine the vector field $u(x)$ and the tensor field $\sigma(x)$, which satisfy Eqs (1.9) and (1.10) with boundary conditions (2.1). We will assume that Ω and Γ_σ are such that the second Korn inequality is satisfied, for example, Ω is a bounded domain which satisfies the cone condition, and Γ_σ is a set, open in Γ [2.4]. We will assume that

$$\sigma_0\in L_2(\Omega),\quad \nabla\sigma_0, f\in L_2(\Omega),\quad q\in L_2(\Gamma_\sigma),\quad a, b\in L_\infty(\Omega)$$

adopting for simplicity the standard notation for corresponding spaces of scalar, vector and tensor functions, where the tensor functions a and b are positive definite uniformly in Ω, i.e. constant $a_0 > 0$ and $b_0 > 0$ exist for which $|\varepsilon|_a^2 \geqslant a_0\varepsilon : \varepsilon$ nearly everywhere in the domain Ω and for any ε.

In this case the tensor function $a - a : d^{-1} : a \in L_\infty(\Omega)$ is uniformly positive definite. Actually, the tensors of the coefficients a and b, specified almost everywhere in Ω, form linear transformations over the six-dimensional space of second-rank tensors. The matrices of these transformations are symmetrical in each space basis and are positive definite. In the special basis $\varepsilon_1, \dots, \varepsilon_6$ of eigentensors of the form

$$b : \varepsilon_k = \beta_k a : \varepsilon_k, \quad \varepsilon_k : a : \varepsilon_k = 1, \quad \varepsilon_j : a : \varepsilon_k = 0 \quad (j \neq k)$$

the matrix of the transformation a is the identity matrix and b is the diagonal matrix with positive eigenvalues $\beta_1, \dots, \beta_6$ along the main diagonal. The easily verified equations

$$(a - a : d^{-1} : a) : \varepsilon_k = \frac{\beta_k}{1+\beta_k} a : \varepsilon_k$$

hold, by means of which the inequality

$$\varepsilon : (a - a : d^{-1} : a) : \varepsilon \geq \beta_0 |\varepsilon|_a^2/(1+\beta_0) \quad (\beta_0 = \min_k \beta_k)$$

is proved by expanding the arbitrary tensor ε in the basis. This inequality enables us to establish the required estimates for the potentials $\varphi(\varepsilon)$ and $\psi(\sigma)$. Since, by virtue of Eq. (1.8)

$$2\varphi \geq |\varepsilon|_a^2 - |a : \varepsilon - \sigma_0|_{d^{-1}}^2 = |\varepsilon|_a^2 - |a : \varepsilon|_{d^{-1}}^2 + 2\varepsilon : a : d^{-1} : \sigma_0 - |\sigma_0|_{d^{-1}}^2$$

then, applying the obvious inequality

$$2\varepsilon : a : d^{-1} : \sigma_0 \geq -\beta|\varepsilon|_a^2 - |d^{-1} : \sigma_0|_a^2/\beta \quad (\beta > 0)$$

it can be shown that

$$|\varepsilon|_a^2 \geq 2\varphi(\varepsilon) \geq (1/(1+\beta_0) - \beta)|\varepsilon|_a^2 - |d^{-1} : \sigma_0|_a^2/\beta - |\sigma_0|_{d^{-1}}^2 \tag{2.2}$$

The chain of inequalities

$$|\sigma|_{a^{-1}}^2 \leq 2\psi(\sigma) \leq |\sigma|_{a^{-1}}^2 + |\sigma - \sigma_0|_{b^{-1}}^2 \tag{2.3}$$

is satisfied for the function $\psi(\sigma)$, defined by expression (1.7).

Generally speaking, not only the tensors of the moduli of elasticity but also the cones $x \in \Omega$ and C depend on the point K in heterogeneous media. We will further assume that in this case the functions

$$x \mapsto \varphi(x, \varepsilon), \quad x \mapsto \psi(x, \sigma)$$

belong to the space $L_\infty(\Omega)$ for any ε and σ.

The assumptions made enable us to establish the solvability of the problem, which leads to two independent variational problems. The unknown displacement field is obtained as a result of minimizing the integral

$$I(u) = \int_\Omega \Big(\varphi(\varepsilon(u)) - f \cdot u\Big) d\Omega - \int_{\Gamma_\sigma} q \cdot u d\Gamma \tag{2.4}$$

in the linear space U of the generalized functions $u \in H^1(\Omega)$, which satisfy boundary condition (2.1) on Γ_u. When determining the stress field, the integral

$$J(\sigma) = \int_\Omega \psi(\sigma) d\Omega \tag{2.5}$$

is minimized in the affine space Σ of the tensor functions $\sigma \in L_2(\Omega)$, for which the equilibrium equations (1.10) and boundary conditions (2.1) on Γ_σ are satisfied. Both integrals are strictly convex functionals with a weakly semi-continuous lower bound. Estimates (2.2) and (2.3) ensure the coerciveness of these functionals. The spaces U and Σ are closed. On the basis of a well-known theorem (see [2, 4–6]) a conclusion can be drawn regarding the existence and uniqueness of the solution. The classical theory of duality [4] establishes a connection between the displacement and stress fields obtained which agrees with the constitutive equations (1.1).

In the important case $\Gamma_u = \varnothing$, when static boundary conditions are given along the entire boundary, the displacement field is necessarily non-unique and is determined, apart from rigid displacements

$$H = \{u | u(x) = w + \omega \cdot x\}$$

where w is an arbitrary vector, ω is a skew-symmetric second-rank tensor. The space $H^1(\Omega)$ is split into the direct sum H and the orthogonal subspace

$$H^\perp(\Omega) = \left\{u \in H^1(\Omega) | \int_\Omega u \, d\Omega = \int_\Omega x \times u \, d\Omega = 0\right\}$$

If Ω is the Lipschitz domain, the third Korn inequality is satisfied. This inequality, by virtue of estimate (2.2) for $\beta < 1/(1 + \beta_0)$ denotes the coerciveness of the functional $I(u)$ in the given subspace and guarantees the existence of a solution when the main vector and main moment of the forces vanish

$$\int_\Omega f d\Omega + \int_\Gamma q \, d\Gamma = \int_\Omega x \times f d\Omega + \int_\Gamma x \times q \, d\Gamma = 0 \tag{2.6}$$

The assertion of the existence and uniqueness of the solution remains in force in the case of heterogeneous boundary conditions in displacements: $u = u_0$ on Γ_u if $u_0 \in H^{1/2}(\Gamma_u)$.

The problem of the solvability of the boundary-value problems within the framework of limit models turns out to be more difficult. The problem in stresses, leading to the minimization of the quadratic functional

$$J_a(\sigma) = \frac{1}{2}\int_\Omega |\sigma|_a^2 \, d\Omega$$

in the convex and closed set

$$\Sigma_K = \{\sigma \in \Sigma | \sigma - \sigma_0 \in K \text{ almost everywhere in } \Omega\}$$

is well posed as $b \to 0$. If the set Σ_K is not empty, the minimum point exists and is unique.

The proof of the existence of a displacement field requires special constructions, since the functional $I_a(u)$, defined by formula (2.4) in terms of $\varphi_a(\varepsilon)$ is not coercive in $H^1(\Omega)$ (see [4]). The uniqueness theorem does not hold.

The problem in displacements is well posed as $a \to \infty$. It reduces to the determination of the minimum point of the quadratic functional

$$I_b(u) = \int_\Omega \left(\frac{1}{2}|\varepsilon(u)|_b^2 + \sigma_0 : \varepsilon(u) - f \cdot u\right) d\Omega - \int_{\Gamma_\sigma} q \cdot u \, d\Gamma \tag{2.7}$$

on the convex and closed cone

$$U_C = \{u \in U | \varepsilon(u) \in C \text{ almost everywhere in } \Omega$$

If $\Gamma_u = \varnothing$, the cone U_C must be considered as the subset $H^\perp(\Omega)$, requiring in addition that conditions (2.6) must be satisfied. The solution of the problem of the minimum exists and is unique (apart from rigid displacements). The construction of the stresses is related to the minimization of the non-coercive functional $J_b(\sigma)$ on Σ. This functional is obtained from (2.5) by replacing $\psi(\sigma)$ by $\psi_b(\sigma)$. In the case when $\Sigma_K \neq \varnothing$, when, as will be shown below, the medium is in an absolutely rigid state, each tensor field $\sigma \in \Sigma_K$ satisfies the equation $J_b(\sigma) = 0$ and is the required minimum point. Hence, the stress field, generally speaking, is non-unique. The proof of the existence of the solution of the problem in stresses is not, in generals, of practical interest due to the obvious mechanical ill-posed nature of the model.

3. THE LIMIT STATES

The domain Ω is split into two parts – a rigid domain, in which the material does not deform, and a domain of non-zero deformation, according to the model for the equilibrium of the medium. The applied external load (f, q) is said to be safe, if there is no deformation zone [7]. In this case the displacement vector or its projection onto $H^{\perp}(\Omega)$ is equal to zero nearly everywhere in Ω.

Let Σ_K be a non-empty set. Then by Green's formula for any $\tilde{\sigma} \in \Sigma_K$ and $\tilde{u} \in U_C$

$$\int_\Omega (\nabla \cdot \tilde{\sigma} + f) \cdot \tilde{u}\, d\Omega = \int_\Omega (f \cdot \tilde{u} - \tilde{\sigma} : \varepsilon(\tilde{u}))\, d\Omega + \int_{\Gamma_\sigma} q \cdot \tilde{u}\, d\Gamma = 0$$

Hence, we obtain

$$W(\tilde{u}) = \int_\Omega (f \cdot \tilde{u} - \sigma_0 : \varepsilon(\tilde{u}))\, d\Omega + \int_{\Gamma_\sigma} q \cdot \tilde{u}\, d\Gamma \leq 0 \tag{3.1}$$

taking into account that $(\tilde{\sigma} - \sigma_0) : \varepsilon(\tilde{u}) \leqslant 0$. The actual displacement field is obtained as the solution of the problem of minimizing the functional $I_b(u)$ on the cone U_C. The point $0 \in U_C$ serves as the vertex of this cone, hence

$$I_b(u) = \min_{\tilde{u} \in U_C} I_b(\tilde{u}) = \min_{\tilde{u} \in U_C} \min_{\lambda \geq 0} I_b(\lambda \tilde{u})$$

The direct calculation of the minimum with respect to λ, taking into account expression (2.7), results in maximization problem

$$I_b(u) = -\max_{\tilde{u} \in U_C} \frac{W_+^2(\tilde{u})}{\int_\Omega |\varepsilon(\tilde{u})|_b^2 d\Omega} \tag{3.2}$$

where $W_+ = (W + |W|)/2$ is the positive part of the expression.

In condition (3.1) is disturbed for a certain element $\tilde{u} \in U_C$, then, judging by the value of the functional $I_b(u)$ is non-zero (strictly negative). To satisfy this condition, $I_b(u) = 0$ for all elements; consequently the unique solution of the minimization problem (2.7) is identically equal to zero. Hence in the case when $\Sigma_K \neq \varnothing$ the load f, q is safe.

By using a version of the principle of duality, given, for example, in [8], the inverse proposition can be proved in the following weak form: if the acting load is safe, then a sequence of tensor functions $\sigma_n \in \Sigma$ exists, for which $\sigma_n - \sigma_0$ tends to the cone K as $n \to \infty$. This means that the sequence of projections onto the conjugate cone C formed by the first terms of the expansions

$$\sigma_n - \sigma_0 = b : \pi_b(b^{-1} : (\sigma_n - \sigma_0)) + \Pi_{b^{-1}}(\sigma_n - \sigma_0)$$

which follow from formula (1.3), tends to zero almost everywhere in the domain Ω. If σ_n is a convergent sequence in $L_2(\Omega)$, then the set Σ_K contains is limit and, consequently, is not empty. However, generally speaking, the limits do not have to exist, but even in this case the existence of such a sequence guarantees the satisfaction of condition (3.1).

Actually, the problem of determining the exact lower bound of the functional

$$J_b(\sigma) = \frac{1}{2}\int_\Omega |\pi_b(b^{-1} : (\sigma - \sigma_0))|_b^2 d\Omega$$

in the affine space Σ due to a shift can easily be reduced to the analogous problem for the linear subspace $\Sigma_0 = \Sigma - \Sigma \subset L_2(\Omega)$. The functional considered is bounded from below on Σ_0 and is continuous, for example, at the point $0 \in \Sigma_0$. As was shown in a more general formulation in [4], the minimization problem (2.7) is a dual problem. Hence all the conditions of the principle of duality are satisfied and from its statement we have the equation

$$\inf_{\tilde{\sigma} \in \Sigma} J_b(\tilde{\sigma}) = -I_b(u) \tag{3.3}$$

in which inf cannot be reached on elements from Σ, while the right-hand side terms out to vanish in view of the safety of the load.

Let σ_n be the minimizing sequence, i.e. $J_b(\sigma_n) \to 0$. Then $\pi_b(b^{-1} : (\sigma_n - \sigma_0)) \to 0$ in $L_2(\Omega)$. As is well known, from the sequence which converges in $L_2(\Omega)$ it is possible to select a subsequence which converges nearly everywhere. This subsequence obviously satisfies the required property. The fact that its limit, if one exists, is part of the set Σ_K is proved taking into account expansions (1.3).

To prove the assertion that condition (3.1) is satisfied we will assume that $\sigma_n \in \Sigma$ is a sequence for which $\sigma_n - \sigma_0$ tends to the cone K. Then, since convergence nearly everywhere results in convergence with respect to the norm $L_2(\Omega)$, the sequence $J_b(\sigma_n)$ tends to zero and is a minimizing sequence, and moreover, the lower bound in (3.3) vanishes. Hence, $I_b(u) = 0$ and $u = 0$.

Thus, the satisfaction of two mutually equivalent conditions: the conditions $\Sigma_K \neq \varnothing$ in the weak form described above and (3.1), is necessary and sufficient for the safety of the load (f, q). The criterion obtained constitutes the content of the static and kinematic theorems of the theory of limiting equilibrium, well developed for models of rigid-plastic media [7, 8].

Safe loads form the convex and closed set S in the Cartesian product of the spaces $L_2(\Omega)$ and $L_2(\Gamma_\sigma)$. The convexity is proved starting from the definition: if (f, q) and $(\tilde{f}, \tilde{q})$ are elements of S, then condition (3.1) is satisfied for them. This condition obviously also holds for the convex combination of loads $(f, q) + (1 - \lambda)(\tilde{f}, \tilde{q})$ with parameter $\lambda \in (0, 1)$. The closure S is a consequence of the continuous dependence of the solution of minimization problem (2.7) on f and q. It is well known [2] that this problem is equivalent to the variational inequality $(u, \tilde{u} \in U_C)$.

$$\int_\Omega \left\{ (b : \varepsilon(u) + \sigma_0) : (\varepsilon(\tilde{u}) - \varepsilon(u)) - f \cdot (\tilde{u} - u) \right\} d\Omega - \int_{\Gamma_\sigma} q \cdot (\tilde{u} - u) d\Gamma \geq 0 \tag{3.4}$$

Assuming here the solution corresponding to the external load $(\tilde{f}, \tilde{q})$ as an arbitrary variable function $\tilde{u}$ and adding (3.4) to the variational inequality characterizing the solution $\tilde{u}$, in which the variable function is equal to u, we can obtain

$$\int_\Omega |\varepsilon(\tilde{u}) - \varepsilon(u)|_b^2 d\Omega \leq \int_\Omega (\tilde{f} - f) \cdot (\tilde{u} - u) d\Omega + \int_{\Gamma_\sigma} (\tilde{q} - q) \cdot (\tilde{u} - u) d\Gamma$$

Hence, applying Korn's inequality to the left-hand and the normative inequalities to the right-hand side, we obtain an estimate which guarantees the continuous dependence of the solution

$$\alpha \|\tilde{u} - u\|_{H^1(\Omega)} \leq \|\tilde{f} - f\|_{L_2(\Omega)} + \|\tilde{q} - q\|_{H^{-1/2}(\Omega)} \quad (\alpha > 0) \tag{3.5}$$

Boundary points of the set S correspond to the limiting loads. We can determine the safety factors – non-negative numbers m and n, for which the load mf, nq is limiting, for any not necessarily safe load (f, q). If the coefficient m is given, where $(mf, 0) \in S$, then by virtue of criterion (3.1), we have

$$n(m) = \inf_{\tilde{u} \in U_C} \left\{ V(\tilde{u}, m) / \left(\int_{\Gamma_\sigma} q \cdot \tilde{u} d\Gamma \right)_+ \right\}, \quad V = \int_\Omega (\sigma_0 : \varepsilon(\tilde{u}) - mf \cdot \tilde{u}) d\Omega \tag{3.6}$$

The function $n(m)$ is concave (Fig. 3), since the exact lower bound of the sum is greater than or equal to the sum of exact lower bounds:

$$n(\lambda m + (1 - \lambda)\tilde{m}) = \inf_{\tilde{u} \in U_C} \left\{ (\lambda V(\tilde{u}, m) + (1 - \lambda) V(\tilde{u}, \tilde{m})) / \left(\int_{\Gamma_\sigma} q \cdot \tilde{u} d\Gamma \right)_+ \right\} \geq$$
$$\geq \lambda n(m) + (1 - \lambda) n(\tilde{m})$$

The characteristic points of intersection of the graph of this function with the coordinate axes in the (m, n) plane can be obtained using relation (3.6) as

$$n_0 = n(0), \quad m_0 = \inf_{\tilde{u} \in U_C} \left\{ \int_\Omega \sigma_0 : \varepsilon(\tilde{u}) d\Omega / \left(\int_\Omega f \cdot \tilde{u} d\Omega \right)_+ \right\}$$

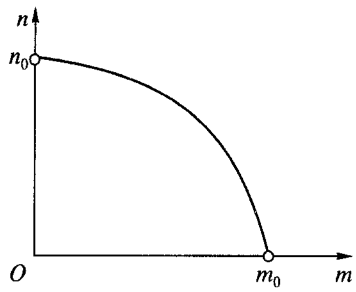

Fig. 3

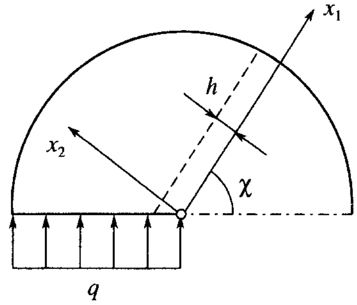

Fig. 4

A similar relation between the safety factors was first analysed for the model of the viscous rigid-plastic medium in [9].

4. LOCALIZATION OF THE DEFORMATIONS

The formulae obtained above provide us with a simple method of estimating the safety factors of the load. As an example we will consider the plane strain state for a homogeneous cylindrical sample of radius r with a radial notch whose sides are loaded with a pressure $q > 0$, caused, for example, thermal expansion of a thin metal plate inserted into the notch. We will describe the different strengths of the material using the Mises–Schleicher condition. According to this condition

$$\sigma_0 = (\tau_0/\kappa)\delta, \quad K = \{\sigma | \tau(\sigma) \le \kappa p(\sigma)\}$$

with τ_0 is the coupling coefficient for simple shear, κ is the coefficient of internal friction, $\rho = -\sigma : \delta/3$ is the hydrostatic pressure and $\tau = \sqrt{\sigma' : \sigma'/2}$ is the intensity of shear stresses (δ is the unit tensor and $\sigma' = \sigma + p\delta$ is the stress deviator). The conjugate cone is

$$C = \{\varepsilon | \gamma(\varepsilon) \le \theta(\varepsilon)/\kappa\}$$

where $\theta = \varepsilon : \delta$ is the deformation of the volume and $\gamma = \sqrt{2\varepsilon' : \varepsilon'}$ is the shear intensity.

Let $\tilde{u}(x)$ be the permissible displacement field describing the localization of the deformation of the simple shear with dilatation in a narrow linear zone of width h, inclined at an angle χ to the line of the notch (Fig. 4). In a Cartesian system of coordinates connected to this zone

$$\tilde{u}_1 = \gamma_0 x_2, \quad \tilde{u}_2 = \varepsilon_0 x_2 \quad (0 \le x_2 \le h) \tag{4.1}$$

The displacements are constant and continuous in the remaining part, outside the zone of localization. The condition $\tilde{u} \in U_C$ takes the form $\gamma_0 \leqslant \varepsilon_0 \sqrt{1/\kappa^2 - 4/3}$ and only makes sense in the case when $\kappa \leqslant \sqrt{3}/2$. The limit pressure is calculated from the formula

$$q = \frac{\tau_0}{\kappa} \inf_{\tilde{u} \in U_C} \left\{ \int_{\Gamma} \tilde{u} \cdot \nu d\Gamma \Big/ \left(-\int_{\Gamma_\sigma} \tilde{u} \cdot \nu d\Gamma \right)_+ \right\}$$

obtained from relation (3.6) using Green's formula. The upper bound of the form

$$q^* = \frac{\tau_0}{\kappa} \frac{\varepsilon_0}{\gamma_0 \sin\chi + \varepsilon_0 \cos\chi}$$

can be obtained by letting h tend to zero. The parameters occurring here must be chosen from the condition for a minimum of q^*. Consequently

$$q^* = \frac{\tau_0}{\sqrt{1 - \kappa^2/3}}, \quad \chi = \operatorname{arctg} \sqrt{\frac{1}{\kappa^2} - \frac{4}{3}} \tag{4.2}$$

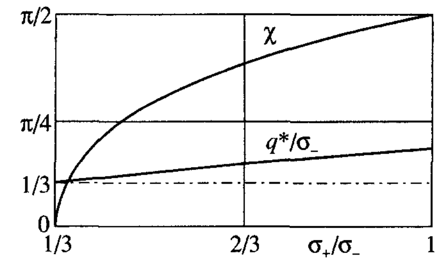

Fig. 5

This is the best upper bound of the pressure and the most probable angle of emergence of the zone of localization of the deformation of the type considered. In the limit as $\kappa \to 0$ localization occurs in a direction perpendicular to the notch. The zone of localization rotates and becomes an extension of the notch for $\kappa \to \sqrt{3}/2$. In the case when $\kappa > \sqrt{3}/2$ uniaxial tension of the medium is inadmissible, and hence the upper bound is constructed by a more complex method and is not included here.

For the majority of natural and artificial materials of different strengths the tensile strength σ_+ is less than the compression strength σ_-. Some of them, for example graphite, are characterized by the fact that the ratio σ_+/σ_- varies practically over the entire range from 0 to 1 depending upon the grade of the material. According to the Mises–Schleicher condition the strength for the uniaxial stressed state is

$$\sigma_\pm = \frac{3\tau_0}{\sqrt{3} \pm \kappa}$$

The coefficient of internal friction is expressed in terms of the ratio of the strengths as

$$0 < \kappa = \sqrt{3}\frac{1 - \sigma_+/\sigma_-}{1 + \sigma_+/\sigma_-} < \sqrt{3}$$

The dimensionless dependences of the angle χ and the quantity q^*/σ on the parameter σ_+/σ_-, obtained from formulae (4.2), are given in Fig. 5.

In conclusion we note that it is not possible to obtain a lower bound of the pressure, close to (4.2), using the static theorem, therefore it is not clear by how much q^* differs from the limiting value. However, numerical calculations of a similar problem [10] showed satisfactory agreement of the results.

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